PERGAMON

International Journal of Solids and Structures 38 (2001) 3073–3079

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Wave properties of some periodic structures

Edoardo Scarpetta [a,*], Mezhlum A. Sumbatyan [b]

[a] *DIIMA, University of Salerno, Via Ponte Don Melillo, 84084 Fisciano (SA), Italy*

[b] *Research Institute of Mechanics and Applied Mathematics, Stachki Prospect 200/1, Rostov-on-Don 344090, Russian Federation*

Received 20 October 1999; in revised form 8 May 2000

Abstract

In the context of wave propagation in damaged (elastic) solids, we develop an analytical approach for normal penetration of a plane wave through a two-dimensional array of cracks. Differently from our previous papers, the cracks' lines are not equally spaced along the direction of propagation (the cracks being periodically distributed only in the orthogonal direction). The linear system analytically obtained by means of a uniform approximation for one-mode range, is submitted to a standard method for numerical resolution. Reflecting the physical intuition, the transmission coefficient turns out to be (almost monotonically) decreasing with distance through the structure. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Periodic structures; Wave properties

1. Introduction

In a foregoing paper (Scarpetta and Sumbatyan, 1997), we studied wave propagation in damaged elastic solids from a fully analytical point of view. By using a uniform approximation in one-mode range previously obtained (Scarpetta and Sumbatyan, 1995), we derived explicit analytical results for all relevant parameters connected with normal penetration of a plane wave through an elastic continuum with a regular, doubly periodic, distribution of *cracks*. Several numerical tests were also performed to estimate the validity of the main assumptions and approximations. In that paper, we noted that such *well-organized* structures exhibit specific properties with respect to wave propagation: for relatively small frequencies, both the transmission and reflection coefficients are periodic, both with respect to the distance along the structure and to the frequency itself. However, more consistently with the wave attenuation which is actually observed in experiments, for increasing frequencies (above certain critical values depending on the geometrical parameters), the transmission coefficient turns out to decay rapidly (exponentially) with respect to the two parameters above. The periodic properties described in Scarpetta and Sumbatyan (1997) seem to be closely connected with the regularity of the flaws distribution, more precisely with the assumption of an

* Corresponding author. Fax: +390-89-965226.
E-mail address: scarpett@diima.unisa.it (E. Scarpetta).

PII: S0020-7683(00)00220-1

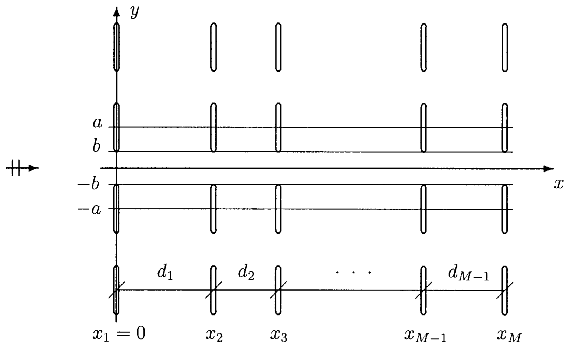

Fig. 1. Plane wave propagation through a semi-periodic array of cracks.

equal distance between the arrays of co-planar cracks (Scarpetta and Sumbatyan, 1997, (Fig. 1)). In this paper, we intend to generalize the previous context, in order to cover the case of different distances; unfortunately, we will not reach fully analytical results as in that paper, but nevertheless we will go deep into a numerical analysis and resolution of the main governing equations. The results will appear to confirm what is well-known from the literature on the subject, namely, the fall of transmission at any frequency when the latticed structure loses certain geometrical regularities. In this connection, the paper of Castanier and Pierre (1995) can be usefully referred to (along with the references therein cited) for a thorough discussion both on the phenomenological aspects of the matter and on the other, (different from ours) numerical/analytical methods of approach. A brief survey of the papers devoted to the study of full-randomly cracked solids can be found in Zhang and Gross (1993); for the same item regarding the regularly (periodically) cracked ones, (Scarpetta and Sumbatyan, 1997) and the references therein cited.

For convenience, we only consider the anti-plane propagation problem in (two-dimensional) elastic context; however, the results can as well apply to similar problems in acoustics and in electromagnetism, according to the interpretation of the wave field in concern.

2. Mathematical formulation

We consider an unbounded elastic medium in which there is a *semi-periodic* distribution of flaws (Fig. 1): this consists of an arbitrary number $M(\geqslant 2)$ of identical (vertical) planes, each of them containing an infinite periodic array of co-planar cracks. The period of a vertical array is $2a$, and the opening between two neighbouring cracks is $2b$ (around $|y| = 0, 2a, 4a, \ldots$).

By contrast with Scarpetta and Sumbatyan (1997), the distances between the cracked planes are supposed to be different from each other: we denote them by d_i, $i = 1, \ldots, M-1$, so that the locations of the planes now become $x_1 = 0$ and $x_m = \sum_{i=1}^{m-1} d_i$, $m = 2, \ldots, M$.

In this context, the *anti-plane problem* for normal penetration of a harmonic plane wave means that an incident (z-polarized) wave of the form $\exp[\mathrm{i}(kx - \omega t)]$ is entering from $-\infty$ into the structure, giving rise throughout to a diffracted (stationary) wave field $\varphi(x, y)$ that satisfies the Helmholtz equation

$$(\partial_{xx} + \partial_{yy})\varphi + k^2\varphi = 0, \tag{2.1}$$

for a given wave number $k = \omega/c$ (c is the transverse wave speed of the material in concern).

By virtue of the natural symmetry and periodicity, we can restrict the problem to a single layer $|y| < a$ with openings $|y| < b$; also, the following representations can be given for the wave field in the various regions:

$$\varphi_l = \mathrm{e}^{\mathrm{i}kx} + R\mathrm{e}^{-\mathrm{i}kx} + \sum_{n=1}^{\infty} A_n \mathrm{e}^{q_n x} \cos(\pi n y/a), \quad x < 0, \tag{2.2a}$$

$$\varphi_m = B_0^m \cos[k(x - x_m)] + C_0^m \cos[k(x - x_{m+1})] + \sum_{n=1}^{\infty} \left\{ B_n^m \mathrm{ch}[q_n(x - x_m)] + C_n^m \mathrm{ch}[q_n(x - x_{m+1})] \right\} \cos(\pi n y/a), \tag{2.2b}$$

$$x_m < x < x_{m+1}, \quad m = 1, \ldots, M-1,$$

$$\varphi_r = T\mathrm{e}^{\mathrm{i}k(x - x_M)} + \sum_{n=1}^{\infty} D_n \mathrm{e}^{-q_n(x - x_M)} \cos(\pi n y/a), \quad x > x_M, \tag{2.2c}$$

where all capital letters denote unknown constants, ch (or sh)-hyperbolic cosine (or sine), and

$$q_n = \sqrt{(\pi n/a)^2 - k^2}, \quad n = 1, 2, \ldots \tag{2.3}$$

Like in Scarpetta and Sumbatyan (1997), we accept the following two main assumptions:

(a) Only *one-mode* propagation is considered, namely $0 < ka < \pi$, so that $q_n > 0\ \forall n$, and at large distances from the structure only plane waves with the given wave number k are present.

(b) The vertical cracked planes are sufficiently distant from each other, so that all ratios d_m/a, $m = 1, \ldots, M-1$, are comparatively large. In practice, it is sufficient for them to be greater enough than a unit value ((Scarpetta and Sumbatyan, 1997) Section 4).

In view of assumption (a), constants R and T in Eqs. (2.2a) and (2.2c) can be fully interpreted as *reflection* and *transmission* coefficients, respectively.

The cracks' faces cannot sustain tangential stress, which is proportional to $\partial\varphi/\partial x$; so, as a natural boundary condition, we can put $\partial\varphi/\partial x = 0$ for $b < |y| < a$ and $x = x_1, \ldots, x_M$. Assuming also the continuity of $\partial\varphi/\partial x$ through the openings, i.e. for $|y| < b$ and $x = x_1, \ldots, x_M$, we can introduce some new unknown functions $g_1(y), \ldots, g_M(y)$, physically related to the stress components along the openings, as follows:

$$\frac{\partial\varphi_l}{\partial x} = \left\{ \begin{array}{ll} g_1(y), & |y| < b, \\ 0, & b < |y| < a \end{array} \right\} = \frac{\partial\varphi_1}{\partial x}, \quad x = x_1 = 0, \tag{2.4a}$$

$$\frac{\partial\varphi_{m-1}}{\partial x} = \left\{ \begin{array}{ll} g_m(y), & |y| < b, \\ 0, & b < |y| < a \end{array} \right\} = \frac{\partial\varphi_m}{\partial x}, \quad x = x_m \quad (m = 2, \ldots, M-1), \tag{2.4b}$$

$$\frac{\partial\varphi_{M-1}}{\partial x} = \left\{ \begin{array}{ll} g_M(y), & |y| < b, \\ 0, & b < |y| < a \end{array} \right\} = \frac{\partial\varphi_r}{\partial x}, \quad x = x_M. \tag{2.4c}$$

The geometrical symmetry implies of course that all these functions be even. By integration of equations above, over $|y| < a$, we easily get

$$\mathrm{i}k(1 - R) = \frac{1}{2a} \int_{-b}^{b} g_1(t)\mathrm{d}t = kC_0^1 \sin(kd_1), \tag{2.5a}$$

$$kC_0^m \sin(kd_m) = \frac{1}{2a} \int_{-b}^{b} g_m(t)\mathrm{d}t = -kB_0^{m-1} \sin(kd_{m-1}) \quad (m = 2, \ldots, M-1), \tag{2.5b}$$

$$\mathrm{i}kT = \frac{1}{2a} \int_{-b}^{b} g_M(t)\mathrm{d}t = -kB_0^{M-1} \sin(kd_{M-1}). \tag{2.5c}$$

Repeating the integration after multiplying by $\cos(\pi n' y/a), n' = 1,2,3,\ldots,$ gives (by orthogonality of cosines)

$$q_n A_n = \frac{1}{a}\int_{-b}^{b} g_1(t)\cos(\pi n t/a)\mathrm{d}t = -C_n^1 q_n \operatorname{sh}(q_n d_1), \tag{2.6a}$$

$$-q_n C_n^m \operatorname{sh}(q_n d_m) = \frac{1}{a}\int_{-b}^{b} g_m(t)\cos\frac{\pi n t}{a}\mathrm{d}t = q_n B_n^{m-1}\operatorname{sh}(q_n d_{m-1}) \quad (m = 2,\ldots,M-1), \tag{2.6b}$$

$$-q_n D_n = \frac{1}{a}\int_{-b}^{b} g_M(t)\cos(\pi n t/a)\mathrm{d}t = B_n^{M-1} q_n \operatorname{sh}(q_n d_{M-1}). \tag{2.6c}$$

Now, the continuity assumption for the wave fields through the openings: $\varphi_l = \varphi_1$ at $x = x_1$, $\varphi_{m-1} = \varphi_m$ at $x = x_m$ $(m = 2,\ldots,M-1)$, $\varphi_{M-1} = \varphi_r$ at $x = x_M$, for $|y| < b$, implies the following equalities:

$$1 + R + \sum_{n=1}^{\infty} A_n \cos(\pi n y/a) = B_0^1 + C_0^1\cos(kd_1) + \sum_{n=1}^{\infty}\left[B_n^1 + C_n^1\operatorname{ch}(q_n d_1)\right]\cos(\pi n y/a), \tag{2.7a}$$

$$\begin{aligned} &B_0^m + C_0^m\cos(kd_m) + \sum_{n=1}^{\infty}\left[B_n^m + C_n^m\operatorname{ch}(q_n d_m)\right]\cos(\pi n y/a) \\ &\quad = B_0^{m-1}\cos(kd_{m-1}) + C_0^{m-1} + \sum_{n=1}^{\infty}\left[B_n^{m-1}\operatorname{ch}(q_n d_{m-1}) + C_n^{m-1}\right]\cos(\pi n y/a) \quad (m = 2,\ldots,M-1), \end{aligned} \tag{2.7b}$$

$$\begin{aligned} &B_0^{M-1}\cos(kd_{M-1}) + C_0^{M-1} + \sum_{n=1}^{\infty}\left[B_n^{M-1}\operatorname{ch}(q_n d_{M-1}) + C_n^{M-1}\right]\cos(\pi n y/a) \\ &\quad = T + \sum_{n=1}^{\infty} D_n\cos(\pi n y/a). \end{aligned} \tag{2.7c}$$

By the main assumptions (a) and (b), we can put $q_n d_m \approx \pi n d_m/a \gg 1$, so that $\operatorname{sh}(q_n d_m) \approx \operatorname{ch}(q_n d_m) \gg 1$ in Eqs. (2.6) and (2.7): this enables us to neglect terms B_n^m or C_n^m with respect to terms $C_n^m\operatorname{ch}(q_n d_m)$ or $B_n^m\operatorname{ch}(q_n d_m)$ $(m = 1,\ldots,M-1)$ in the square brackets of Eqs. (2.7) ((Scarpetta and Sumbatyan, 1997) Section 2). By this approximation, inserting the values of all constants, taken from Eqs. (2.5) and (2.6), into Eqs. (2.7a)–(2.7c), gives finally rise to the following square system of integral equations for the unknowns $g_1,\ldots,g_M$ over the interval $|y| < b$ $(ctg = \cos/\sin)$:

$$\frac{1}{a}\int_{-b}^{b} g_1(t)\left[\frac{1}{4ik} + \frac{ctg(kd_1)}{4k} - \sum_{n=1}^{\infty}\frac{1}{q_n}\cos\frac{\pi n(y-t)}{a}\right]\mathrm{d}t - \frac{1}{4ak\sin(kd_1)}\int_{-b}^{b} g_2(t)\mathrm{d}t = 1, \tag{2.8a}$$

$$\begin{aligned} &\frac{1}{a}\int_{-b}^{b} g_m(t)\left[\frac{ctg(kd_m) + ctg(kd_{m-1})}{4k} - \sum_{n=1}^{\infty}\frac{1}{q_n}\cos\frac{\pi n(y-t)}{a}\right]\mathrm{d}t - \frac{1}{4ak\sin(kd_m)}\int_{-b}^{b} g_{m+1}(t)\mathrm{d}t \\ &\quad - \frac{1}{4ak\sin(kd_{m-1})}\int_{-b}^{b} g_{m-1}(t)\mathrm{d}t = 0 \quad (m = 2,\ldots,M-1), \end{aligned} \tag{2.8b}$$

$$-\frac{1}{4ak\sin(kd_{M-1})}\int_{-b}^{b} g_{M-1}(t)\mathrm{d}t + \frac{1}{a}\int_{-b}^{b} g_M(t)\left[\frac{1}{4ik} + \frac{ctg(kd_{M-1})}{4k} - \sum_{n=1}^{\infty}\frac{1}{q_n}\cos\frac{\pi n(y-t)}{a}\right]\mathrm{d}t = 0. \tag{2.8c}$$

If the new (even) unknown function $h(y)$ is introduced, like in Scarpetta and Sumbatyan (1995, 1997), as a solution of the following equation

$$\frac{1}{a}\int_{-b}^{b} h(t)\left[\sum_{n=1}^{\infty}\frac{1}{q_n}\cos\frac{\pi n(y-t)}{a}\right]\mathrm{d}t = 1, \quad |y| < b, \tag{2.9}$$

then, by linearity, we arrive at

$$g_1(y) = \left[\frac{ctg(kd_1) - i}{4ak}G_1 - \frac{G_2}{4ak\sin(kd_1)} - 1\right]h(y), \tag{2.10a}$$

$$g_m(y) = \left[\frac{ctg(kd_m) + ctg(kd_{m-1})}{4ak}G_m - \frac{G_{m-1}}{4ak\sin(kd_{m-1})} - \frac{G_{m+1}}{4ak\sin(kd_m)}\right]h(y) \quad (m = 2, \ldots, M-1), \tag{2.10b}$$

$$g_M(y) = \left[-\frac{G_{M-1}}{4ak\sin(kd_{M-1})} + \frac{ctg(kd_{M-1}) - i}{4ak}G_M\right]h(y), \tag{2.10c}$$

where we have put

$$G_m = \int_{-b}^{b} g_m(t)\mathrm{d}t, \quad m = 1, \ldots, M. \tag{2.11}$$

Put also $H = \int_{-b}^{b} h(t)\mathrm{d}t$. Integration of Eqs. (2.10a)–(2.10c) over $|y| < b$ finally yields the following square system of linear algebraic equations for the unknowns $G_1, \ldots, G_M$:

$$\left(1 - \frac{ctg(kd_1) - i}{4ak}H\right)G_1 + \frac{H}{4ak\sin(kd_1)}G_2 = -H, \tag{2.12a}$$

$$\frac{H}{4ak\sin(kd_{m-1})}G_{m-1} + \left(1 - \frac{ctg(kd_m) + ctg(kd_{m-1})}{4ak}H\right)G_m + \frac{H}{4ak\sin(kd_m)}G_{m+1} = 0, \quad (m = 2, \ldots, M-1), \tag{2.12b}$$

$$\frac{H}{4ak\sin(kd_{M-1})}G_{M-1} + \left(1 - \frac{ctg(kd_{M-1}) - i}{4ak}H\right)G_M = 0. \tag{2.12c}$$

As clearly explained in Scarpetta and Sumbatyan (1995, 1997), the function $h(y)$ can be explicitly calculated from Eq. (2.9) by means of a uniform one-mode approximation ($q_n \approx \pi n/a$ for $n \geqslant 2$). For its integral, we got in those papers

$$H = -\frac{\{\pi/\ln[\sin(\pi b/2a)]\}[1 - (1 - \pi/aq_1)\sin^4(\pi b/2a)]}{1 - (1 - \pi/aq_1)\{\sin^4(\pi b/2a) - \cos^4(\pi b/2a)/\ln[\sin(\pi b/2a)]\}}, \tag{2.13}$$

so that constant H in system (2.12) is actually a known quantity.

3. Numerical simulation and conclusions

If all distances d_m are equal to each other ($d_m = d$, $\forall m$: "well-organized" structure), then the system (2.12) can be resolved explicitly (Scarpetta and Sumbatyan, 1997). In that case, for not too high frequencies,

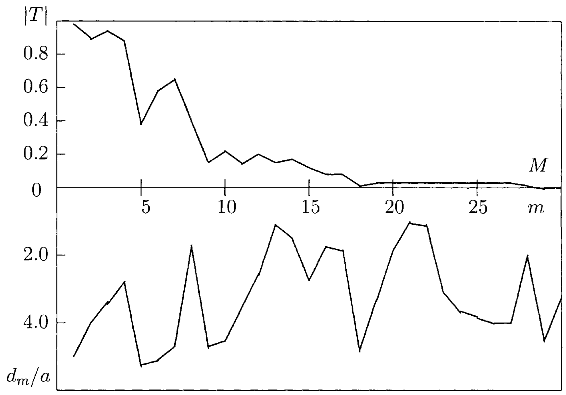

Fig. 2. Transmission coefficient $|T|$ versus number of arrays M for a random distribution of d_m/a, $m = 1, \ldots, M-1$, on the interval $(1.0, 6.0)$: $b/a = 0.5$ and $ak = 0.5$.

both the reflection and transmission coefficients are periodic functions with respect to the number M of vertical arrays (representing the distance):

$$R^{-1} = \left(1 + \frac{iH}{2ak}\right)\{1 - e^{ikd}[\cos\alpha - \sin\alpha\, ctg(M\alpha)]\},$$
$$T = R\frac{iH\sin\alpha}{2ak\sin(M\alpha)}, \qquad \cos\alpha = \cos(kd) - \frac{2ak}{H}\sin(kd). \tag{3.1}$$

This periodic property is not confirmed by experiments, which actually show a monotonic (typically, exponential) decrease of T with distance.

For arbitrary values of d_m, the linear algebraic system (2.12) contains only three main diagonals, being symmetric with respect to the main diagonal. So, a standard "sweep method" may be applied to solve this system numerically (Malcolm and Palmer, 1974).

To achieve more realistic physical properties of the damaged medium, we performed a numerical treatment of the system (2.12) for random values of d_m/a uniformly distributed over some interval (A,B). The transmission coefficient, that is given by Eq. (2.5c) as $T = G_M/2iak$, turns out to vanish with parameter M increasing, for arbitrary (fixed) values of all other physical and geometrical parameters. A typical case is reflected in Fig. 2, where the values of d_m/a are randomly taken in the interval $(1.0, 6.0)$. The lower curve just gives the respective values of d_m/a for $m = 1, \ldots, M-1$. [1]

Let us formulate the principal physical conclusions. As noted above, "well-organized" structures (where the cracks create some quite regular geometric lattice) provide a periodic dependence of the transmission coefficient with distance. Absolutely random damaged structures, which are those typically tested in experiments, provide a monotonically (exponentially) vanishing transmission coefficient. As a consequence, the geometry concerned in the present paper could be considered as a "semi-organized" structure, since the transmission coefficient actually tends to zero with distance, but its decreasing does not appear strictly monotonic.

[1] For instance, the nine random values of d_m/a when $M = 10$ are the ordinates of the nine corner-points (including the initial one) in the lower curve before $M = 10$.

References

Castanier, M.P., Pierre, C., 1995. Lyapunov exponents and localization phenomena in multi-coupled nearly periodic systems. J. Sound Vibr. 183, 493–515.

Malcolm, M.A., Palmer, J., 1974. A fast method for solving a class of tridiagonal linear systems. ACM 17, 14–17.

Scarpetta, E., Sumbatyan, M.A., 1995. Explicit analytical results for one-mode normal reflection and transmission by a periodic array of screens. J. Math. Anal. Appl. 195, 736–749.

Scarpetta, E., Sumbatyan, M.A., 1997. On wave propagation in elastic solids with a doubly periodic array of cracks. Wave Motion 25, 61–72.

Zhang, Ch., Gross, D., 1993. Wave attenuation and dispersion in randomly cracked solids – I, II. Int. J. Engng. Sci. 31, 841–872.